Velocity distribution for flow in pressurized pipes: A critical review

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Abstract. This paper provides a critical review of velocity distribution in fluid flows within pipes. Models developed from the classical deterministic approach and Principle of Maximum Entropy (PME) are presented and discussed. According to the deterministic approach, the velocity distribution in laminar flow depends on the rheological model of the fluid. For turbulent flows, the well be known velocity profiles proposed by Prandtl-von Kárman lack physical consistency across the entire pipe region. The power-law model depends on the Reynolds number and fluid viscosity, while also exhibiting physical inconsistencies. On the other hand, entropic velocity distribution (EVD) was obtained from the PME, which in turn is based on the information theory, Shannon entropy (SE), random variable and constraints related to total probability and the conservation laws of mass, momentum, and energy. Thus, EVD represents a conceptual and generalized model that is physically consistent and satisfies all the assumptions required for flow within pipes. Additionally, it does not necessitate prior knowledge of the fluid rheological model and can be applied to both Newtonian and non-Newtonian fluids regardless of the rheological model, flow regime and roughness of the pipe.

Keywords. Velocity distribution, information theory, maximum entropy.

1. Introduction

Fluid flows in pressurized pipes are of great interest for engineering. The physical analysis of such flows requires knowledge of the velocity and shear stress distributions within the pipe. The model for the shear stress distribution is generalized since it is derived from a balance of forces on a fluid element and is independent of the rheological behavior and flow regime.

In contrast, concerning the velocity distribution, there is no universally generalized and absolutely consistent model obtained solely from a deterministic approach. It has been well known that the proposed models have several limitations, such as:

i. In a laminar flow regime, the velocity distribution depends on the rheological model of the fluid;

ii. For turbulent flow, the well-known logarithmic profile proposed by Prandtl-von Kárman does not satisfy the basic premises of flow such as maximum velocity at the center of the pipe, null velocity at the wall, null velocity gradient at the center and nonnull velocity gradient at the wall (Chiu et al., 1993; Louzada et al, 2021). Furthermore, the parameters of the model must be determined experimentally;

iii. The power-law model also exhibits physical inconsistencies and depends on the Reynolds number and, consequently, the viscosity of the fluid;

iv. In his investigations into the influence of pipe roughness, Nikuradse proposed an empirical model developed from studies with Newtonian fluid.

Therefore, there is a clear necessity for developing a generalized, conceptual, and physically consistent model for velocity distribution in pipes. In their investigations, Chiu (1988) and Chiu et al. (1993) developed theoretical models applicable to both open channels and pressurized pipes based on the Principle of Maximum Entropy (PME) that overcomes the aforementioned limitations. PME is a variational principle that relies on information theory, random variables, Shannon entropy, Lagrangian function, total probability, and the laws of conservation of mass, momentum, and energy (Chiu, 1989). The entropic velocity distribution

(EVD) rigorously satisfies the aforementioned premises and overcomes all limitations of models obtained through the deterministic approach (Souza and Moraes, 2017; Louzada et al., 2021).

2. Rheological Models

Rheology studies the flow and deformation of matter and allows for the determination of the relationships between shear stress (τ), shear rate ($\dot{\gamma}$), and viscosity (μ) (Slatter, 1997; Eshtiaghi et al. 2013).

Shear stress is the shearing force per unit area responsible for the flow of a fluid. Shear rate corresponds to the displacement of fluid elements relative to the distance between them. Viscosity is the property of the fluid that reflects its resistance to flow. Apparent viscosity encompasses not only its inherent resistance to motion but also its flow state (Agwu et al. 2021).

Newtonian fluids are those that exhibit a linear relationship between shear stress and shear rate, as demonstrated by Eq. (1) (Sedaghat, 2017).

$$\tau = \mu.\dot{\gamma} = \mu.\left(\frac{du}{dr}\right) \quad (1)$$

All fluids that do not obey Eq. (1) are classified as non-Newtonian.

A large number of rheological models for non-Newtonian fluids have been proposed in the literature (Kelessidis and Maglione, 2006; Eshtiaghi et al. 2013; Bharathan et al. 2019; Agwu et al., 2021). However, for the purposes of this work, only the three most common non-Newtonian models (Table 1) will be discussed. The Power Law model has the K and n parameters. The first one is named fluid consistency index and indicates the fluid's resistance to flow. The second parameter is the flow behavior index which indicates how close a particular fluid is to Newtonian behavior. Thus, the fluid will be Newtonian if n=1, shear thinning when n<1.0, and shear thickening if n>1.0. The Bingham model includes a parameter called yield stress (τ_o), which corresponds to the minimum shear stress required to initiate the flow of the fluid. The Herschel-Bulkley rheological model has the parameters K, n, and τ_o.

Tab. 1 - Rheological models of non-Newtonian fluids.

Rheological Model	Constitutive Equation
Power-law	$\tau = K(\dot{\gamma})^n$
Bingham	$\tau = \eta.(\dot{\gamma}) + \tau_o$
Herschel-Bulkley	$\tau = K(\dot{\gamma})^n + \tau_o$

3. Classical Approach

For flows in pipes, the distribution of shear stress (τ_{rz}) results from the balance of forces acting on a fluid element and is defined as (Lam et al. 2007)

$$\tau_{rz} = \left(-\frac{\Delta P}{L}\right).\left(\frac{r}{2}\right) \quad (2)$$

According to the classical approach of fluid mechanics, the velocity profile in laminar flow is obtained from the shear stress distribution and the rheological model of the fluid, as expressed in Eq. (3).

$$Shear\ stress\ distribution = Rheological\ model\ of\ the\ fluid \quad (3)$$

The differential equation (DE) that results from Eq. (3), should be solved by applying a boundary condition i.e. $u(R)$ = 0.

The simplest case is the laminar flow of a Newtonian fluid and for this specific condition, DE is derived from Eqs. (1) and (2) and can be written as

$$\left(-\frac{\Delta P}{L}\right).\left(\frac{r}{2}\right) = -\mu\left(\frac{du}{dr}\right) \quad (4)$$

By solving Eq. (4) it is possible to obtain the classical velocity distribution for a Newtonian fluid defined by

$$u(r) = \left(-\frac{\Delta P}{L}\right).\frac{R^2}{4\mu}\left[1-\left(\frac{r}{R}\right)^2\right] \quad (5)$$

For flows involving Bingham fluids, Eq. (3) becomes Eq. (6), from which the model represented by Eq. (7) is obtained (Chhabra and Richardson, 1999).

$$\left(-\frac{\Delta P}{L}\right).\left(\frac{r}{2}\right) = -\mu\left(\frac{du_z}{dr}\right) + \tau_o \quad (6)$$

$$u(r) = \left(-\frac{\Delta P}{L}\right)\frac{R^2}{4\mu}\left[1-\left(\frac{r}{R}\right)^2\right] - \frac{\tau_o}{\mu}R\left(1-\frac{r}{R}\right) \quad (7)$$

For non-Newtonian fluids that obey the power-law, the DE is found to be

$$\left(-\frac{\Delta P}{L}\right).\left(\frac{r}{2}\right) = K\left(\frac{du}{dy}\right)^n \quad (8)$$

In this case, the velocity profile derived from the power-law rheological model is defined as

$$u(r) = \left(-\frac{\Delta P.R}{2KL}\right)^{\frac{1}{n}}.\left(\frac{n}{n+1}\right).R.\left[1-\left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right] \quad (9)$$

From this discussion, it is clear that for laminar flow regime, the velocity profiles can only be obtained if the rheological model of the fluid is previously known. In other words, each constitutive rheological equation provides its own velocity distribution.

Likewise, there are also limitations associated with the velocity distributions for turbulent regimes. Turbulent flow in pipes is classically approached by considering different flow regions. The discussion presented by Chhabra and Richardson (1999) is based on the following regions: viscous sublayer, transition region, and turbulent zone. On the other hand, Bird et al. (2004) address turbulent flow based on four coexisting regions within the pipe:

i. Viscous Sublayer: this is a region near the pipe wall where viscous effects dominate. The classical approach to turbulent flow in pipes is based on the assumption that the velocity profile is linear in the viscous sublayer;

ii. Buffer Layer: Located between the viscous and inertial sublayers, where both viscous and turbulent effects are significant;

iii. Inertial Sublayer: Near the main turbulent flow;

iv. Turbulent Core: The region that encompasses the majority of the flow and vortices.

For turbulent flow regimes, the Newtonian rheological model should consider both dynamic viscosity (μ) and turbulence-associated kinematic viscosity (v_t), as depicted by Eq. (10) (Chhabra and Richardson, 1999).

$$\tau = \left(\frac{\mu}{\rho} + v_t\right)\frac{d(\rho\mu)}{dy} \tag{10}$$

According to Prandtl's postulate, turbulent kinematic viscosity depends on a dimension l, referred to as the mixing length, and it maintains a linear relationship with the distance from the wall (y). Thus, kinematic viscosity and the mixing length are defined based on Eqs (11) and (12), respectively.

$$v_t = l^2\frac{du}{dy} \tag{11}$$

$$l = ky \tag{12}$$

Substituting Eqs. (11) and (12) into Eq. (10) yields Eq. (13), expressed in terms of the wall shear stress (τ_w) (Chhabra and Richardson, 1999).

$$\tau_w\left(\frac{R-y}{R}\right) = \left[\frac{\mu}{\rho} + k^2y^2\left(\frac{du}{dy}\right)\right]\frac{d(\rho u)}{dy} \tag{13}$$

Eq. (13) will be simplified when one of the effects (viscous or turbulence) dominates over the other. Due to the proximity of the viscous sublayer to the wall, viscous effects prevail over those arising from turbulence, rendering the latter negligible. Thus, Eq. (13) becomes

$$\tau_w = \left(\frac{\mu}{\rho}\right)\frac{d(\rho u)}{dy} \tag{14}$$

The velocity profile obtained from Eq. (14) is given by

$$u(y) = \frac{\tau_w}{\mu}y \tag{15}$$

Therefore, according to this approach, for Newtonian fluids in the viscous sublayer, the velocity profile is linear. Thus, the approach from Eq. (10) has two clear limitations:

i. Newtonian fluids;

ii. Linear profile velocity in the viscous sublayer.

Eq. (15) can also be expressed as a function of shear velocity (u^*), dimensionless velocity (u^+), and dimensionless length (y^+). Shear velocity is defined from the wall shear stress (τ_w) and density of the fluid (ρ), as

$$u^* = \sqrt{\frac{\tau_w}{\rho}} \tag{16}$$

Dimensionless velocity and length are defined based on Eqs. (17) and (18), respectively (Peker; Helvaci, 2008).

$$u^+ = \frac{u}{u^*} \tag{17}$$

$$y^+ = \frac{y\rho(u^*)}{\mu} \tag{18}$$

By substituting Eqs. (17) and (18) into Eq. (15), one obtains

$$u^+u^* = \left(\frac{\tau_w}{\mu}\right)\left(\frac{y^+\mu}{\rho u^*}\right) \tag{19}$$

In the region where most of the turbulent flow occurs ($y \ll R$), turbulence effects dominate over viscous effects and, consequently, Eq. (13) reduces to

$$\frac{\tau_w}{\rho} = k^2y^2\left(\frac{du}{dy}\right)^2 \tag{20}$$

The velocity profile in terms of shear velocity (u^*) is obtained by substituting Eq. (16) into Eq. (20), leading to

$$u^* = ky\frac{du}{dy} \tag{21}$$

By solving the DE one gets Eq. (22), which can be expressed in terms of dimensionless velocity (u^+) and dimensionless distance (y^+) respectively, as shown in Eq. (23) (McKeon, 2004).

$$\frac{u}{u^*} = \frac{1}{k}\,ln\,ln\,y + C \tag{22}$$

$$u^+ = A_1\,ln\,ln\,y^+ + A_2 \tag{23}$$

Eq. (23) is the well-known logarithmic velocity distribution of Prandtl-von Kárman, where A_1 and A_2 are parameters that need to be determined experimentally. Another limitation that should be considered is that both parameters depend on the flow field in a pipe (transition and turbulent), as can be seen from Tab. 2 (Bird et al. 2004; Peker and Helvaci, 2008).

Tab. 2 - Velocity profiles corresponding to different regions of turbulent flow.

Flow regime	Condition	Speed profile
Viscous sublayer	$\frac{yu^*\rho}{\mu} < 5$	$\frac{u}{u^*} = \frac{yu^*\rho}{\mu}$
Transition	$5 < \frac{yu^*\rho}{\mu} < 30$	$\frac{u}{u^*} = 5{,}0\,ln\left(\frac{yu^*\rho}{\mu}\right) - 3{,}05$
Turbulent	$\frac{yu^*\rho}{\mu} > 30$	$\frac{u}{u^*} = 2{,}5\,ln\left(\frac{yu^*\rho}{\mu}\right) + 5{,}5$

Several researchers have devoted their works to studying the influence of roughness on fluid flow in pressurized pipes. In the 1930s, using creative and innovative techniques, Nikuradse conducted experiments in pipes covered with sand grains with an average diameter of 0.8 mm in order to produce a rough surface. By using water as the working fluid, Nikuradse was able to achieve highly turbulent flows and proposed an empirical model for the mean velocity defined according to Eq. (24) (Chiu et al.

1993).

$$\frac{\bar{u}}{u_{max}} = 1{,}17\left[1 + 9{,}02\left(\frac{u}{u^*}\right)^{-1}\right]^{-1} \quad (24)$$

Where $\bar{u}$, u_{max} and u^* are the mean, maximum, and shear velocities, respectively. Eq. (24) is an empirical and non-generalized model since it was developed from experiments with a Newtonian fluid.

The power-law model for the velocity profile in turbulent flows, represented by Eq. (25), is also frequently reported in the literature (Fox et al. 2006). The parameter a, which depends on the Reynolds number (Re), is defined by Eq. (26).

$$\frac{u}{u_{max}} = \left(1 - \frac{r}{R}\right)^{\frac{1}{a}} \quad (25)$$

$$a = -1{,}7 + 1{,}8 log Re \quad (26)$$

This model also has physical inconsistencies, as the velocity gradient is infinite at the wall and nonnull at the center of the pipe (Fox et al. 2006). Furthermore, it depends on the Reynolds number and, therefore, requires prior knowledge of the fluid viscosity.

While the physical inconsistencies do not prevent the use of the equations for practical engineering purposes, the aforementioned models do not provide high accuracy across the entire cross-sectional area of the pipe.

4. Approach based on the Maximum Entropy Principle

Unlike the purely deterministic approach, the PME is based on Information Theory and encompasses both the deterministic and probabilistic domains (Louzada et al. 2021). The entropic velocity distribution (EVD) modeled by Chiu et al. (1993) was obtained through the Shannon entropy (SE), random variable (velocity), two constraints (total probability and conservation of mass), and maximization of SE by means of the Lagrange multipliers method (Singh, 2011). The SE for the random variable, $H(u)$, is defined as a function of probability density, $f(u)$, as

$$H(u) = -\int_0^{u_{max}} f(u) \ln f(u)\, du \quad (27)$$

The maximization method based on the Lagrangian function ($\mathcal{L}$) requires the definition of constraints. The first one, associated with total probability (C_0) and the second two related to mass conservation (C_1), defined by Eqs. (28) and (29), respectively, must be considered (Chiu, 1987).

$$C_0 = \int_0^{u_{max}} f(u) du = 1 \quad (28)$$

$$C_1 = \int_0^{u_{max}} u f(u) du = \bar{u} \quad (29)$$

In this way, the Lagrangian function is defined as

$$\mathcal{L} = -\int_0^{u_{max}} f(u) \ln[f(u)]\, du + \lambda_1\left[\int_0^{u_{max}} f(u) du - 1\right] + \lambda_2\left[\int_0^{u_{max}} u f(u) du - \bar{u}\right] \quad (30)$$

Where λ_1 and λ_2 are the Lagrange multipliers associated with constraints C_0 and C_1, respectively. The maximization method requires the differentiation of the Lagrangian function with respect to the probability density function according to Eq. (31).

$$\frac{\partial L}{\partial f(u)} = -\{1 + \ln[f(u)]\} + \lambda_1 + \lambda_2 u = 0 \quad (31)$$

Eq. (31) allows to obtain $f(u)$, defined as

$$f(u) = e^{[(\lambda_1 - 1) + \lambda_2 u]} \quad (32)$$

The Lagrange multipliers λ_1 and λ_2 are determined by means of the constraints. Thus, by substituting Eq. (32) into Eq. (28), one gets

$$e^{(\lambda_1 - 1)} = \frac{\lambda_2}{[e^{(\lambda_2 u_{max})} - 1]} \quad (33)$$

The equation relating the mean and maximum velocities is obtained from the second constraint by substituting Eq. (32) into Eq. (29). The result is

$$\int_0^{u_{max}} u f(u) du = \int_0^{u_{max}} u\, e^{[(\lambda_1 - 1) + \lambda_2 u]} du = e^{(\lambda_1 - 1)} \int_0^{u_{max}} u e^{\lambda_2 u} du = \bar{u} \quad (34)$$

After integrating Eq. (34), it is possible to obtain an expression for the mean velocity as a function of λ_1 and λ_2, according to Eq. (35).

$$\bar{u} = e^{(\lambda_1 - 1)}\left\{\frac{u_{max} e^{(\lambda_2 u_{max})}}{\lambda_2} - \frac{[e^{(\lambda_2 u_{max})} - 1]}{{\lambda_2}^2}\right\} \quad (35)$$

The mean velocity can be expressed solely as a function of λ_2 by substituting (33) into (35). The result is

$$\bar{u} = \frac{u_{max} e^{(\lambda_2 u_{max})}}{[e^{(\lambda_2 u_{max})} - 1]} - \frac{1}{\lambda_2} \quad (36)$$

Chiu et al. (1993) defined the product ($\lambda_2 u_{max}$) as an entropy parameter (M), which allows modeling the velocity profile and indicates the degree of turbulence in a flow. Thus, Eq. (36) becomes

$$\frac{\bar{u}}{u_{max}} = \frac{e^M}{e^M - 1} - \frac{1}{M} \quad (37)$$

Chiu et al. (1993) used dimensionless coordinates ξ, ξ_o , and ξ_{max} to define the cumulative velocity distribution, $F(u)$, as

$$F(u) = \left(\frac{\xi - \xi_o}{\xi_{max} - \xi_o}\right) = \int_0^u f(u) du \quad (38)$$

The expression that relates $f(u)$ and $F(u)$ is

$$f(u) = \frac{dF(u)}{du} = \frac{\left(\frac{d\xi}{du}\right)}{(\xi_{max} - \xi_o)} \quad (39)$$

Eq. (32) in Eq. (39), one obtains

$$\left(\frac{d\xi}{du}\right) = \left(\xi_{max} - \xi_o\right) e^{[(\lambda_1 - 1) + \lambda_2 u]} \quad (40)$$

Eq. (40) is solved by separating variables and integrating them, as indicated by Eq. (41).

$$\frac{\xi}{(\xi_{max}-\xi_o)} = e^{(\lambda_1-1)}\left(\frac{e^{\lambda_2 u}}{\lambda_2}\right) + C \quad (41)$$

When $u = 0$, $\xi = \xi_o$, and from this boundary condition, it is possible to obtain the velocity distribution as

$$u = \left(\frac{1}{\lambda_2}\right) ln\left[1 + \frac{\lambda_2}{e^{(\lambda_1-1)}}\left(\frac{\xi-\xi_o}{\xi_{max}-\xi_o}\right)\right] \quad (42)$$

It is more convenient to express Eq. (42) solely as a function of λ_2. Thus, by substituting Eq. (33) with Eq. (42), one obtains

$$u = \left(\frac{1}{\lambda_2}\right) ln\left[1 + [e^{(\lambda_2 u_{max})} - 1]\left(\frac{\xi-\xi_o}{\xi_{max}-\xi_o}\right)\right] \quad (43)$$

The EVD as a function of curvilinear coordinates and entropy parameter is defined according to Eq. (44) (Chiu et al. 1993).

$$u_E(\xi) = \left(\frac{u_{max}}{M}\right) ln\left[1 + [e^M - 1]\left(\frac{\xi-\xi_o}{\xi_{max}-\xi_o}\right)\right] \quad (44)$$

Finally, EVD can also be written as a function of radial distance (r) and pipe radius (R) as shown in Eq. (45) (Singh, 2014; Louzada et al. 2021).

$$u_E(r) = \left(\frac{u_{max}}{M}\right) ln\left[1 + [e^M - 1]\left(1 - \frac{r^2}{R^2}\right)\right] \quad (45)$$

Where:

$$\xi = \frac{\pi R^2 - \pi r^2}{\pi R^2},\ \xi_o = 0\ and\ \xi_{max} = 1$$

Chiu (1988) defines the entropic parameter as a measure of the uniformity of probability and velocity distributions. Furthermore, M is a parameter that models the velocity profile and indicates the degree of turbulence in a flow. Thus, the flow will be laminar if $M = 0$ and turbulent when $M > 0$.

With regard to EVD defined by Eq. (45), the velocity is maximum at the pipe axis and zero at the wall i.e. $u(0) = u_{max}$ and $u(R) = 0$. In order to prove the physical consistency of the EVD, the limit can be applied when M tends to zero according to Eq. (46).

$$\lim_{M\to 0}\left(\frac{u_{max}}{M}\right) ln\left\{1 + [e^M - 1]\left[1 - \left(\frac{r}{R}\right)^2\right]\right\} = \frac{0}{0} \quad (46)$$

Applying the L'Hopital's rule and taking the limit again, Eq. (46) becomes the classical velocity profile for the laminar flow regime, represented by Eq. (47).

$$u = u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right] \quad (47)$$

The EVD is a generalized model because it is independent of the type of fluid (Newtonian or non-Newtonian), rheological model, flow regime (laminar and turbulent), and pipe roughness. Furthermore, it satisfies all the premises required in a flow.

In addition to the physical consistency and generalization of EVD, it must be mentioned its relevant application in rheology, since a model for entropic shear rate (ESR) can be obtained from Eqs. (37) and (45), as presented in Eq. 48 (Louzada et al. 2021).

$$\dot{\gamma}_w = \left(\frac{8\underline{u}}{D}\right)\left[\frac{(e^M - 1)^2}{2(Me^M - e^M + 1)}\right] \quad (48)$$

Obviously, the ESR model can also be employed for Newtonian and non-Newtonian fluids regardless of the rheological model, flow type, and roughness of the pipe.

5. Conclusion

The classical and deterministic approach to obtaining velocity distribution in pipe flow has several limitations. As pointed out earlier, in laminar flow the velocity distribution depends on the rheological model of the fluid i.e. each rheological equation results in a specific velocity profile. In the case of turbulent flow, the model proposed by Prandtl-von Kárman does not satisfy all the required premises for the flow and does not provide accuracy in all regions of the pipe. The logarithmic velocity profile has parameters that can only be determined experimentally. The power-law model, besides displaying physical inconsistency, depends on the Reynolds number and, therefore, requires prior knowledge of the fluid viscosity. On the other hand, the EVD overcomes the aforementioned limitations as it is a generalized model that can be employed for any fluid (Newtonian and non-Newtonian), regardless of its rheological model, flow regime, and roughness of the pipe.

Notation

The following symbols are used in the paper:

A_1 e A_2 = parameters of the Prandtl-von Kárman logarithmic velocity distribution;
C_0= constraint related to total probability;
C_1 = constraint related to mass conservation;
$F(u)$ = cumulative velocity distribution;
$f(u)$ = probability density function of u;
$H(u)$ = entropy of velocity;
K = consistency index;
k = proportionality constant;
l = mixing length;
$\mathcal{L}$ = Lagrangian function;
M = entropy parameter;
n = behavior index;
a = Reynolds number-dependent parameter;
R = inner radius of the tube;
Re= Reynolds number;
r = radial distance;
u = velocity;
u_E entropic velocity distribution;
u_{max} = maximum velocity;
u^* = shear velocity;
$\bar{u}$ = mean velocity;
u^+ = dimensionless velocity;
v_t = kinematic viscosity;
y = distance from the wall;
y^+= dimensionless distance;
$\dot{\gamma}$ = shear rate;
η = apparent viscosity;
λ_1 e λ_2 = Lagrange multipliers;
μ = dynamic viscosity;

$\left(-\frac{\Delta P}{L}\right)$ = pressure gradient;
ξ isovel coordinate;
ξ_o e ξ_{max}= minimum and maximum values of ξ;
ρ = fluid-specific mass;
τ = sher stress;
τ_o = yield stress;
τ_{rz} = stress distribution;
τ_w = wall shear stress.

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